# 9.1 Sequences and Series

Sequences and series model many real-life situations over time. For example, in Exercise 98 on page 619, a sequence models the percent of United States adults who met federal physical activity guidelines from 2007 through 2014.

- **Use sequence notation to write the terms of sequences.**
- **Use factorial notation.**
- **Use summation notation to write sums.**
- **Find the sums of series.**
- **Use sequences and series to model and solve real-life problems.**

## Sequences

In mathematics, the word *sequence* is used in much the same way as in ordinary English. Saying that a collection is listed in *sequence* means that it is ordered so that it has a first member, a second member, a third member, and so on. Two examples are 1, 2, 3, 4, . . . and 1, 3, 5, 7, . . . .

Mathematically, you can think of a sequence as a *function* whose domain is the set of positive integers. Rather than using function notation, however, sequences are usually written using subscript notation, as shown in the following definition.

**Definition of Sequence**

An **infinite sequence** is a function whose domain is the set of positive integers. The function values

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

are the **terms** of the sequence. When the domain of the function consists of the first $n$ positive integers only, the sequence is a **finite sequence.**

On occasion, it is convenient to begin subscripting a sequence with 0 instead of 1 so that the terms of the sequence become

$$a_0, a_1, a_2, a_3, \ldots.$$

When this is the case, the domain includes 0.

**REMARK** The subscripts of a sequence make up the domain of the sequence and serve to identify the positions of terms within the sequence. For example, $a_4$ is the fourth term of the sequence, and $a_n$ is the $n$th term of the sequence. Any variable can be a subscript. The most commonly used variable subscripts in sequence and series notation are $i, j, k,$ and $n$.

**EXAMPLE 1 Writing the Terms of a Sequence**

**a.** The first four terms of the sequence given by $a_n = 3n - 2$ are

$a_1 = 3(1) - 2 = 1$ 1st term

$a_2 = 3(2) - 2 = 4$ 2nd term

$a_3 = 3(3) - 2 = 7$ 3rd term

$a_4 = 3(4) - 2 = 10.$ 4th term

**b.** The first four terms of the sequence given by $a_n = 3 + (-1)^n$ are

$a_1 = 3 + (-1)^1 = 3 - 1 = 2$ 1st term

$a_2 = 3 + (-1)^2 = 3 + 1 = 4$ 2nd term

$a_3 = 3 + (-1)^3 = 3 - 1 = 2$ 3rd term

$a_4 = 3 + (-1)^4 = 3 + 1 = 4.$ 4th term

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Write the first four terms of the sequence given by $a_n = 2n + 1$.

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**REMARK** Write the first four terms of the sequence given by

$$a_n = \frac{(-1)^{n+1}}{2n+1}.$$

Are they the same as the first four terms of the sequence in Example 2? If not, then how do they differ?

## EXAMPLE 2 A Sequence Whose Terms Alternate in Sign

Write the first four terms of the sequence given by $a_n = \dfrac{(-1)^n}{2n+1}$.

**Solution** The first four terms of the sequence are as follows.

$$a_1 = \frac{(-1)^1}{2(1)+1} = \frac{-1}{2+1} = -\frac{1}{3} \qquad \text{1st term}$$

$$a_2 = \frac{(-1)^2}{2(2)+1} = \frac{1}{4+1} = \frac{1}{5} \qquad \text{2nd term}$$

$$a_3 = \frac{(-1)^3}{2(3)+1} = \frac{-1}{6+1} = -\frac{1}{7} \qquad \text{3rd term}$$

$$a_4 = \frac{(-1)^4}{2(4)+1} = \frac{1}{8+1} = \frac{1}{9} \qquad \text{4th term}$$

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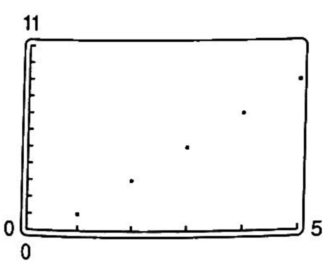
Write the first four terms of the sequence given by $a_n = \dfrac{2+(-1)^n}{n}$.

Simply listing the first few terms is not sufficient to define a unique sequence—the $n$th term *must be given.* To see this, consider the following sequences, both of which have the same first three terms.

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots$$

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{15}, \ldots, \frac{6}{(n+1)(n^2-n+6)}, \ldots$$

**TECHNOLOGY** To graph a sequence using a graphing utility, set the mode to *sequence* and *dot* and enter the expression for $a_n$. The graph of the sequence in Example 3(a) is shown below. To identify the terms, use the *trace* feature or *value* feature.




## EXAMPLE 3 Finding the $n$th Term of a Sequence

Write an expression for the apparent $n$th term $(a_n)$ of each sequence.

**a.** $1, 3, 5, 7, \ldots$ **b.** $2, -5, 10, -17, \ldots$

**Solution**

**a.** $n$: 1 2 3 4 . . . $n$

*Terms*: 1 3 5 7 . . . $a_n$

*Apparent pattern:* Each term is 1 less than twice $n$. So, the apparent $n$th term is

$$a_n = 2n - 1.$$

**b.** $n$: 1 2 3 4 . . . $n$

*Terms*: 2 −5 10 −17 . . . $a_n$

*Apparent pattern:* The absolute value of each term is 1 more than the square of $n$, and the terms have alternating signs, with those in the even positions being negative. So, the apparent $n$th term is

$$a_n = (-1)^{n+1}(n^2+1).$$

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Write an expression for the apparent $n$th term $(a_n)$ of each sequence.

**a.** $1, 5, 9, 13, \ldots$ **b.** $2, -4, 6, -8, \ldots$

Some sequences are defined **recursively.** To define a sequence recursively, you need to be given one or more of the first few terms. All other terms of the sequence are then defined using previous terms.

**EXAMPLE 4** **A Recursive Sequence**

Write the first five terms of the sequence defined recursively as

$a_1 = 3$

$a_k = 2a_{k-1} + 1, \quad \text{where } k \geq 2.$

**Solution**

$a_1 = 3$ — 1st term is given.

$a_2 = 2a_{2-1} + 1 = 2a_1 + 1 = 2(3) + 1 = 7$ — Use recursion formula.

$a_3 = 2a_{3-1} + 1 = 2a_2 + 1 = 2(7) + 1 = 15$ — Use recursion formula.

$a_4 = 2a_{4-1} + 1 = 2a_3 + 1 = 2(15) + 1 = 31$ — Use recursion formula.

$a_5 = 2a_{5-1} + 1 = 2a_4 + 1 = 2(31) + 1 = 63$ — Use recursion formula.

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Write the first five terms of the sequence defined recursively as

$a_1 = 6$

$a_{k+1} = a_k + 1, \quad \text{where} \quad k \geq 1.$

In the next example, you will study a well-known recursive sequence, the Fibonacci sequence.

**EXAMPLE 5** **The Fibonacci Sequence: A Recursive Sequence**

The Fibonacci sequence is defined recursively, as follows.

$a_0 = 1$

$a_1 = 1$

$a_k = a_{k-2} + a_{k-1}, \quad \text{where} \quad k \geq 2$

Write the first six terms of this sequence.

**Solution**

$a_0 = 1$ — 0th term is given.

$a_1 = 1$ — 1st term is given.

$a_2 = a_{2-2} + a_{2-1} = a_0 + a_1 = 1 + 1 = 2$ — Use recursion formula.

$a_3 = a_{3-2} + a_{3-1} = a_1 + a_2 = 1 + 2 = 3$ — Use recursion formula.

$a_4 = a_{4-2} + a_{4-1} = a_2 + a_3 = 2 + 3 = 5$ — Use recursion formula.

$a_5 = a_{5-2} + a_{5-1} = a_3 + a_4 = 3 + 5 = 8$ — Use recursion formula.

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Write the first five terms of the sequence defined recursively as

$a_0 = 1, \quad a_1 = 3, \quad a_k = a_{k-2} + a_{k-1}, \quad \text{where} \quad k \geq 2.$

## Factorial Notation

Many sequences involve terms defined using special products called **factorials.**

> **Definition of Factorial**
>
> If $n$ is a positive integer, then $n$ **factorial** is defined as
>
> $$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1) \cdot n.$$
>
> As a special case, zero factorial is defined as $0! = 1$.

**REMARK** The value of $n$ does not have to be very large before the value of $n!$ becomes extremely large. For example, $10! = 3{,}628{,}800$.

Notice that $1! = 1$, $2! = 1 \cdot 2 = 2$, $3! = 1 \cdot 2 \cdot 3 = 6$, and $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$. Factorials follow the same conventions for order of operations as exponents. So,

$$2n! = 2(n!) = 2(1 \cdot 2 \cdot 3 \cdot 4 \cdots n), \quad \text{whereas} \quad (2n)! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots 2n.$$

### EXAMPLE 6 Writing the Terms of a Sequence Involving Factorials

Write the first five terms of the sequence given by $a_n = \frac{2^n}{n!}$. Begin with $n = 0$.

**Algebraic Solution**

$a_0 = \frac{2^0}{0!} = \frac{1}{1} = 1$ — 0th term

$a_1 = \frac{2^1}{1!} = \frac{2}{1} = 2$ — 1st term

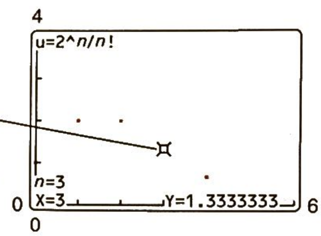
$a_2 = \frac{2^2}{2!} = \frac{4}{2} = 2$ — 2nd term

$a_3 = \frac{2^3}{3!} = \frac{8}{6} = \frac{4}{3}$ — 3rd term

$a_4 = \frac{2^4}{4!} = \frac{16}{24} = \frac{2}{3}$ — 4th term

**Graphical Solution**

Using a graphing utility set to *dot* and *sequence* modes, enter the expression for $a_n$. Next, graph the sequence. Use the graph to estimate the first five terms.




Use the *trace* feature to approximate the first five terms.

$u_0 = 1$

$u_1 = 2$

$u_2 = 2$

$u_3 \approx 1.333 \approx \frac{4}{3}$

$u_4 \approx 0.667 \approx \frac{2}{3}$

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Write the first five terms of the sequence given by $a_n = \frac{3^n + 1}{n!}$. Begin with $n = 0$.

**ALGEBRA HELP** Here is another way to simplify the expression in Example 7(a).

$$\frac{8!}{2! \cdot 6!} = \frac{8 \cdot 7 \cdot \cancel{6!}}{2 \cdot 1 \cdot \cancel{6!}} = 28$$

When fractions involve factorials, you can often divide out common factors.

### EXAMPLE 7 Simplifying Factorial Expressions

**a.** $\frac{8!}{2! \cdot 6!} = \frac{\cancel{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \cdot 7 \cdot 8}{1 \cdot 2 \cdot \cancel{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}} = \frac{7 \cdot 8}{2} = 28$

**b.** $\frac{n!}{(n-1)!} = \frac{\cancel{1 \cdot 2 \cdot 3 \cdots (n-1)} \cdot n}{\cancel{1 \cdot 2 \cdot 3 \cdots (n-1)}} = n$

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Simplify the factorial expression $\frac{4!(n+1)!}{3!n!}$.

▷ **TECHNOLOGY** Most graphing utilities can sum the first $n$ terms of a sequence. Consult the user's guide for your graphing utility for specific instructions on how to do this using the *sum* and *sequence* features or a *series* feature.

## Summation Notation

A convenient notation for the sum of the terms of a finite sequence is called **summation notation** or **sigma notation.** It involves the use of the uppercase Greek letter sigma, written as $\Sigma$.

### Definition of Summation Notation

The sum of the first $n$ terms of a sequence is represented by

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4 + \cdots + a_n$$

where $i$ is the **index of summation,** $n$ is the **upper limit of summation,** and 1 is the **lower limit of summation.**

**REMARK** Summation notation is an instruction to add the terms of a sequence. Note that the upper limit of summation tells you the last term of the sum. Summation notation helps you generate the terms of the sequence prior to finding the sum.

**EXAMPLE 8** **Summation Notation for a Sum**

**a.** $\displaystyle\sum_{i=1}^{5} 3i = 3(1) + 3(2) + 3(3) + 3(4) + 3(5)$

$= 45$

**b.** $\displaystyle\sum_{k=3}^{6} (1 + k^2) = (1 + 3^2) + (1 + 4^2) + (1 + 5^2) + (1 + 6^2)$

$= 10 + 17 + 26 + 37$

$= 90$

**REMARK** In Example 8, note that the lower limit of a summation does not have to be 1 and the index of summation does not have to be the letter $i$. For example, in part (b), the lower limit of summation is 3 and the index of summation is $k$.

**c.** $\displaystyle\sum_{i=0}^{8} \frac{1}{i!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!}$

$\displaystyle = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} + \frac{1}{40{,}320}$

$\approx 2.71828$

For this summation, note that the sum is very close to the irrational number

$e \approx 2.718281828.$

It can be shown that as more terms of the sequence whose $n$th term is $1/n!$ are added, the sum becomes closer and closer to $e$.

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Find the sum $\displaystyle\sum_{i=1}^{4} (4i + 1)$.

### Properties of Sums

1. $\displaystyle\sum_{i=1}^{n} c = cn$, $c$ is a constant.

2. $\displaystyle\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$, $c$ is a constant.

3. $\displaystyle\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

4. $\displaystyle\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

For proofs of these properties, see Proofs in Mathematics on page 686.

## Series

Many applications involve the sum of the terms of a finite or infinite sequence. Such a sum is called a **series.**

### Definition of Series

Consider the infinite sequence $a_1, a_2, a_3, \ldots, a_i, \ldots$

1. The sum of the first $n$ terms of the sequence is called a **finite series** or the **$n$th partial sum** of the sequence and is denoted by

$$a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i.$$

2. The sum of all the terms of the infinite sequence is called an **infinite series** and is denoted by

$$a_1 + a_2 + a_3 + \cdots + a_i + \cdots = \sum_{i=1}^{\infty} a_i.$$

**EXAMPLE 9** **Finding the Sum of a Series**

*See LarsonPrecalculus.com for an interactive version of this type of example.*

For the series

$$\sum_{i=1}^{\infty} \frac{3}{10^i}$$

find (a) the third partial sum and (b) the sum.

**Solution**

**a.** The third partial sum is

$$\sum_{i=1}^{3} \frac{3}{10^i} = \frac{3}{10^1} + \frac{3}{10^2} + \frac{3}{10^3}$$

$$= 0.3 + 0.03 + 0.003$$

$$= 0.333.$$

**b.** The sum of the series is

$$\sum_{i=1}^{\infty} \frac{3}{10^i} = \frac{3}{10^1} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \frac{3}{10^5} + \cdots$$

$$= 0.3 + 0.03 + 0.003 + 0.0003 + 0.00003 + \cdots$$

$$= 0.33333\ldots$$

$$= \frac{1}{3}.$$

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For the series

$$\sum_{i=1}^{\infty} \frac{5}{10^i}$$

find (a) the fourth partial sum and (b) the sum.

Notice in Example 9(b) that the sum of an infinite series can be a finite number.

## Application

Sequences have many applications in business and science. Example 10 illustrates one such application.

**EXAMPLE 10** **Compound Interest**

An investor deposits \$5000 in an account that earns 3% interest compounded quarterly. The balance in the account after $n$ quarters is given by

$$A_n = 5000\left(1 + \frac{0.03}{4}\right)^n, \quad n = 0, 1, 2, \ldots .$$

**a.** Write the first three terms of the sequence.

**b.** Find the balance in the account after 10 years by computing the 40th term of the sequence.

**Solution**

**a.** The first three terms of the sequence are as follows.

$$A_0 = 5000\left(1 + \frac{0.03}{4}\right)^0 = \$5000.00 \qquad \text{Original deposit}$$

$$A_1 = 5000\left(1 + \frac{0.03}{4}\right)^1 = \$5037.50 \qquad \text{First-quarter balance}$$

$$A_2 = 5000\left(1 + \frac{0.03}{4}\right)^2 \approx \$5075.28 \qquad \text{Second-quarter balance}$$

**b.** The 40th term of the sequence is

$$A_{40} = 5000\left(1 + \frac{0.03}{4}\right)^{40} \approx \$6741.74. \qquad \text{Ten-year balance}$$

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An investor deposits \$1000 in an account that earns 3% interest compounded monthly. The balance in the account after $n$ months is given by

$$A_n = 1000\left(1 + \frac{0.03}{12}\right)^n, \quad n = 0, 1, 2, \ldots .$$

**a.** Write the first three terms of the sequence.

**b.** Find the balance in the account after four years by computing the 48th term of the sequence.

## Summarize (Section 9.1)

1. State the definition of a sequence *(page 610)*. For examples of writing the terms of sequences, see Examples 1–5.
2. State the definition of a factorial *(page 613)*. For examples of using factorial notation, see Examples 6 and 7.
3. State the definition of summation notation *(page 614)*. For an example of using summation notation, see Example 8.
4. State the definition of a series *(page 615)*. For an example of finding the sum of a series, see Example 9.
5. Describe an example of how to use a sequence to model and solve a real-life problem *(page 616, Example 10)*.